2.7.1.1 Von Mises-Hen&y (isotropic materials only)

The Von Mises-Hencky or maximum distortion energy theory predicts that a ductile material under a general stress state will yield when its shear distortional energy (the total strain energy minus the strain energy attributable to change in volume) is equal to the shear distortional energy under simple tension (Hertzberg, 1989; Nahas, *1986). This* yield theory is only valid for isotropic materials and, consequently, is not generally appropriate for composite materials. It has shown good correlation with test data for metals under multi-axial loading since it includes the interactive effects of all the stress components. The Von Mises-Hencky theory is applied by first calculating the equivalent or effective stress (σ_e) acting on a material element (Hertzberg, 1989) according to

$$\sigma_e = \frac{\sqrt{2}}{2}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)\right]^{0.5} \qquad (59)$$

This failure theory predicts that the material will fail when the effective stress becomes greater than the material yield strength, σr, or when

$$\sigma_e > \sigma_y \qquad (60)$$

2.7.1.2 Maximum Stress

Perhaps the most widely used failure criterion for unidirectional composites is the maximum stress failure criterion (Tsai, 1987; Nahas, 1986), which predicts that a material will fail when the magnitude of the stress in any direction exceeds its corresponding allowable level in that direction. This criterion is valid for both isotropic and anisotropic materials. However, it does not consider interactions between the various stress components and, therefore, has the potential to be inaccurate for multi-axial stress states. The most significant advantage of this failure criterion is that it identifies the specific mode of failure within a ply. Failure in any principal direction of the material is predicted when any of the following conditions exist:

if $\sigma 1 > 0$ and if $\sigma 1 >$ XlT, then the failure mode is fiber tension (61a)
if $\sigma 1 < 0$ and if $|\sigma 1| >$ XlC, then the failure mode is fiber compression (61b)
if $\sigma 2 > 0$ and if $\sigma 2 >$ X2T, then the failure mode is matrix tension (61c)
if $\sigma 2 < 0$ and if $|\sigma 2| >$ X2C, then the failure mode is matrix compression (61d)
if $\sigma 3 > 0$ and if $\sigma 3 >$ X3T, then the failure mode is matrix tension (61e)
if $\sigma 3 < 0$ and if $|\sigma 3| >$ X3C, then the failure mode is matrix compression (61f)
if $|\sigma 4| >$ X23, then the failure mode is interlaminar shear (61e)
if $|\sigma 5| >$ X13, then the failure mode is interlaminar shear (61f)
if $|\sigma 6| >$ X12, then the failure mode is in-plane shear (61g)

In Equations (61a) through (61g), $\sigma 1$ through $\sigma 6$, are the six principal ply stresses, XlT is the tensile strength in the 1-direction (longitudinal), XlC is the compressive strength in the l-direction, X2T is the tensile strength in the 2-direction (transverse), X2C is the compressive strength in the 2-direction, X3T is the tensile strength in the 3-direction, X3C is the compressive strength in the 3-direction, X23 is the shear strength in the 23-plane, Xl3 is the shear strength in the 13-plane, and Xl2 is the shear strength in the 12-plane.

2.7.1.3 Maximum Strain

The maximum strain failure criterion (Tsai, 1987; Nahas, 1986) predicts that a material will fail when the strain in any direction exceeds its allowable level. This failure criterion is similar to the maximum stress criterion, except that it accounts for some of the interactions between the stresses that are attributable to the Poisson's effects in the material (i.e., stresses in the l- and 3- directions will affect the strain in the 2-direction). The failure criterion is applied in the exact same manner as the maximum stress failure criterion. The mechanical strains in the six directions ($\varepsilon 1$, $\varepsilon 2$, $\varepsilon 3$, $\varepsilon 4$, $\varepsilon 5$ and $\varepsilon 6$) are compared to their corresponding maximum strain allowables (Y1T, YlC, Y2T, Y2C, Y3T, Y3C, Y23, Y13, and Y12) in the same manner as described for the maximum stress criterion, for example,

if $\varepsilon 1>0$ and if $\varepsilon 1>\mathrm{Y1T}$, then the failure mode is fiber tension (62a)
if $\varepsilon 1<0$ and if $|\varepsilon 1|>\mathrm{YlC}$, then the failure mode is fiber compression (62b)
if $\varepsilon 2>0$ and if $\varepsilon 2>\mathrm{Y2T}$, then the failure mode is matrix tension (62c)
if $\varepsilon 2<0$ and if $|\varepsilon 2|>\mathrm{Y2C}$, then the failure mode is matrix compression (62d)
if $\varepsilon 3>0$ and if $\varepsilon 3>\mathrm{Y3T}$, then the failure mode is matrix tension (62e)
if $\varepsilon 3<0$ and if $|\varepsilon 3|>\mathrm{Y3C}$, then the failure mode is matrix compression (62f)
if $|\varepsilon 4|>\mathrm{Y23}$, then the failure mode is interlaminar shear (62e)
if $|\varepsilon 5|>\mathrm{Y13}$, then the failure mode is interlaminar shear (62f)
if $|\varepsilon 6|>\mathrm{Y12}$, then the failure mode is in-plane shear (62g)

2.7.1.4 Hydrostatic Pressure Adjusted

The hydrostatic pressure-adjusted failure criterion is actually a modified version of the maximum stress failure criterion that accounts for stress interactions under compressive loading. The criterion is based on empirical observations reported in the literature (Hahn and Kallas, 1992). The failure criterion essentially states that all the compression strength allowables in the principal directions (XlC, X2C, and X3C) and the shear strength allowables (X23, X13, and X12) increase with the hydrostatic pressure state (HP) that exists within the 23 planes of a ply. A review of this phenomenon is discussed in a separate report (Hoppel, Bogetti, Gillespie 1995).

For the particular version of this criterion presented in this report, several assumptions are made. First, the "hydrostatic pressure state" in the 23 planes referred to here (HP) is not that which is defined in the traditional sense. It is assumed to be equal to either (1) the minimum of $|\sigma 2|$ and $|\sigma 3|$ (when both are compressive), (2) the average of $\sigma 2$ and $\sigma 3$ (when this average is compressive), or (3) zero when either of these two conditions is not satisfied. In addition, tensile strength allowables are assumed to be independent of this hydrostatic pressure influence, and the strength allowables in the 2 and 3 directions are assumed to be equal (ie., X2T=X3T, X2C=X3C). It is further assumed that all the shear strength allowables are equal (ie., S23=S13=S12).

Material stre strength here is assumed to be a bi-linear function with the respect to the hydrostatic pressure state in the 23 planes of a ply. This relationship is expressed explicitly as a function of the assumed hydrostatic pressure state, HP, according to

$$X1C(HP) = X1C(O)+ML\,1 * HP \text{ for } (HP<LTP) \tag{63a}$$

and

$$XlC(HP) = (XlC(O)+ML\,l*LTP)+ML2*HP \text{ for } (HP>LTP) \tag{63b}$$

in which XlC(0) is the usual longitudinal compression strength allowable (i.e., XlC(O) = XlC); ML1 and ML2 are the two slopes describing the bi-linear relationship, and LTP is the hydrostatic pressure state in which a change in slope of the XlC versus HP relationship occurs. Similarly, for the transverse directions,

$$X2C(HP) = X3C(HP) = X2C(O) + MTl\,.\,HP \text{ for } (HP<TTP) \tag{63c}$$

and

$$X2C(HP) = X3C(HP) = (X2C(O) + MT1\,.\,TTP)+MT2*HP \text{ for } (HP>TTP) \tag{63d}$$

in which X2C(O) is the usual transverse compression strength allowable (i.e., X2C(O)=X2C=X3C); MT1 and MT2 are the two slopes describing the bi-linear relationship, and TTP is the hydrostatic pressure state in which a change in slope of the X2C (or X3C) versus HP relationship occurs. For the shear directions, the following relationships are used:

$$X23(HP) = X13(HP) = Xl2(HP) = S+MSl\,.HP \text{ for } (HP<STP) \tag{63e}$$

and

$$X23(HP) = Xl3(HP) = Xl2(HP) = (S+MSl\,.STP)+MS2.HP \text{ for } (HP>STP) \tag{63f}$$

in which S is the usual shear strength allowable (i.e., S=X23=X13=X12); MS1 and MS2 are the two slopes describing the bi-linear relationship, and STP is the hydrostatic pressure state in which a change in slope of the S (or S23 or S13 or S12) versus HP relationship occurs.

2.7.1.5 Tsai-Wu Quadratic Interaction

The Tsai-Wu quadratic interaction or tensor polynomial failure criterion (Tsai and Wu 1971) accounts for the interactive effects of a multi-axial stress state. Failure is predicted when the following condition occurs:

$$\begin{aligned}&F1\sigma1, + F2(\sigma2 + \sigma3) + F11\sigma^2{}_1 + F_{22}(\sigma^2{}_2 + \sigma^2{}_3) + 2F_{12}\sigma1(\sigma2 + \sigma3) + \\ &2F_{23}\sigma_2\sigma_3 + 2F_{44}\sigma^2{}_4 + F_{66}(\sigma^2{}_5 + \sigma^2{}_6) >= 1\end{aligned} \tag{64}$$

in which $\sigma1$ through $\sigma6$ are the principal stresses in the lamina. The constants $F1$, $F2$, F_{11}, $F22$, $F12$, $F23$, $F44$, and $F66$ are defined by the following expressions:

$$F1 = 1/X1T - 1/X1C \tag{65a}$$

$$F2 = 1/X2T - 1/X2C \tag{65b}$$

$$F11 = 1/(X1T)(X1C) \tag{65c}$$

$$F22 = 1/(X2T)(X2C) \tag{65d}$$

$$F44 = 2(F22 - F23) \tag{65e}$$

$$F66 = 1/S^2_{23} = 1/S^2_{13} = 1/S^2_{12} \tag{65f}$$

The constants F_{12} and F_{23} are determined experimentally. Methods to determine these constants are described in Tsai and Wu (1971) and Jiang and Tennyson (1989). This theory assumes that the material is transversely isotropic in the principal 1-2 plane of the composite. The major drawback of this failure criterion is that it does not distinguish among the various potential modes of failure.

2.7.1.6 Christensen's Criterion

Christensen (1988) proposed a strain-based failure criterion, which identifies failure as being either fiber dominated or matrix dominated while considering the multi-axial stress state for matrix-dominated failure. This criterion identifies three distinct failure modes for a composite lamina: fiber tension, fiber compression, and matrix failure. Christensen's failure criterion has been translated into a stress-based failure criterion by Hahn and Kallas (1992), and the stress based failure criterion is employed in this work.

Fiber tension and fiber compression, respectively, are predicted to occur when either Equation 66a or 66b is satisfied.

$$\left|\frac{(YIT)(E_1)}{\sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3}\right| <= 1 \text{ (if } \sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3 > 0) \tag{66a}$$

$$\left|\frac{(YIC)(E_1)}{\sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3}\right| <= 1 \text{ (if } \sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3 < 0) \tag{66b}$$

in which Y1T is the tensile failure strain for the lamina in the 1 direction, Y1C is the compressive failure strain for the lamina in the 1 direction, and E_l is the elastic Young's modulus in the fiber direction. Matrix-dominated failure is predicted to occur when the following condition exists:

$$A\sigma_1 + B(\sigma_2 + \sigma_3) + C\sigma_1^2 + D(\sigma_2^2 + \sigma^2_3) + E\sigma_1(\sigma_2 + \sigma_3) + F\sigma_2\sigma_3 + G\sigma^2_1 + H(\sigma^2_5 + \sigma^2_6) <= 1 \tag{67}$$

in which the coefficients A, B, C, D, E, F, G, and H are given by

$$A = \alpha(1-2\nu_{12}) / K^2E_1 \tag{68}$$

$$B = \alpha(1-\nu_{21} - \nu_{23}) / K^2E_2 \tag{68b}$$

$$C = 2(1+\nu_{12})^2 / 3K^2E^2{}_1 \tag{68c}$$

$$D = 2[(1+\nu_{21}+ \nu^2{}_{21})+(1-\nu_{21}) \nu_{23} + \nu^2{}_{23}] / 3K^2E_2{}^2 \tag{68d}$$

$$E = 2[(-1-2\nu_{21}+ \nu_{23})(1+\nu_{12})]/ 3K^2E_1E_2 \tag{68e}$$

$$F = 2[(-1-2\nu_{21}+ 2\nu^2{}_{21}) - 2(2+\nu_{21}) \nu_{23} - \nu_{31}]/ 3K^2E_1E_2 \tag{68f}$$

$$G = 1/ 2K^2G^2{}_{23} \tag{68g}$$

$$H = 1/ 2K^2G^2{}_{12} \tag{68h}$$

The constants K and α are experimentally determined material parameters, El and E2 are the elastic moduli in the principal and transverse directions, G12 and G23 are the in-plane and out-of-plane shear moduli, respectively, and the ν_{ij}'s are the usual Poisson's ratios for the lamina.

2.7.1.7 Feng's Failure Criterion

Feng (1991) also proposed a strain-based failure criterion that differentiates between fiber-dominated and matrix-dominated failure under multi-axial loading. This criterion determines failure, based on the strain invariants in the lamina. Matrix-dominated failure is predicted to occur when the following relation exists:

$$A_1J_l + A_{ll}J_1{}^2 + A_2J_2 - 1 >= 0 \tag{69}$$

in which A_1, $A_{11}$and A2 are empirically determined parameters and J_1 and J_2 are the strain invariants given by

$$J_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{70a}$$

$$J_2 = \{[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_3 - \varepsilon_2)^2 + (\varepsilon_1 - \varepsilon_3)^2] / 6) + \varepsilon_4{}^2 + \varepsilon_5{}^2 + \varepsilon_6{}^2 \tag{70b}$$

in which ε_1, ε_2, $\varepsilon_{3,}$ ε_4, ε_5, and ε_6 are the ply strains. Fiber-dominated failure is predicted to occur when the following is satisfied:

$$A_5J_5 + A_{55}J_5{}^2 + A_4J_4 - 1 >= 0 \tag{71}$$

in which $A_{5,}$ A_{55}, and A_4 are experimentally determined parameters, and J_4 and J_5 are the strain invariants given by

$$J_4 = \varepsilon_4^2 + \varepsilon_5^2 \tag{72a}$$
$$J_5 = \varepsilon_1 \tag{72b}$$

2.7.1.8 Modified Hashin’s Failure Criterion

Hashin (1980) proposed a stress-based failure criterion for composite materials, which considers the tri-axial stress state for matrix failure modes but only considers the uni-axial stress state in the fiber direction for fiber-dominated failure modes. In the fiber direction, this criterion is the same as the maximum stress failure criterion. Gipple, Nuismer, and Camponeschi (1995) suggested modifications of the failure criterion proposed by Hashin in which they assume that compressive matrix failure occurs because of a shearing mechanism rather than through compression. The modified Hashin’s failure criterion is currently being used by several government agencies and contractors and is presented in this report.

For the fiber-dominated failure modes, the modified Hashin failure criterion distinguishes between tension and compression according to the following conditions:

if $\sigma_1>0$ and if $\sigma_1 >= X1T$, then the failure mode is fiber tension (73a)
if $\sigma_1<0$ and if $|\sigma_1| >= XlC$, then the failure mode is fiber compression (73b)

Matrix-dominated failure is predicted when either of the following two conditions exists:

$$(\sigma_{nm}/X2T)^2 + (\sigma_{nt}/X2C)^2 + (\sigma_{nl}/X12)^2 \geq 1 \quad (\text{for } \sigma_{nm}>0 \text{ (matrix tension)}) \tag{74a}$$
$$(\sigma_{nt}/X2C)^2 + (\sigma_{nl}/X12)^2 \geq 1 \quad (\text{for } \sigma_{nm}<0 \text{ (matrix compression)}) \tag{74b}$$

in which the normal (σ_{nm}), normal-transverse (σ_{nt}) and normal-longitudinal (σ_{nl}) stresses are evaluated according to the following:

$$\sigma_{nm} = (\sigma_2 + \sigma_3 / 2) + (\sigma_2 - \sigma_3 / 2)\cos(2\beta) + \sigma_{23}\sin(2\beta) \tag{75a}$$
$$\sigma_{nt} = (\sigma_2 - \sigma_3 / 2)\sin(2\beta) + \sigma_{23}\cos(2\beta) \tag{75b}$$
$$\sigma_{nl} = (\sigma_{13} - \sigma_3 / 2)\sin(2\beta) + \sigma_{23}\cos(2\beta) \tag{75c}$$

Here, the angle β defines an orientation in the 2-3 plane in which the maximum matrix stress state in the lamina exists.